Fully developed laminar flow and heat transfer in smooth trapezoidal microchannel

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Abstract

Numerical analysis of fully developed laminar slip flow and heat transfer in trapezoidal micro-channels has been studied with uniform wall heat flux boundary conditions. Through coordinate transformation, the governing equations are transformed from physical plane to computational domain, and the resulting equations are solved by a finite-difference scheme. The influences of velocity slip and temperature jump on friction coefficient and Nusselt number are investigated in detail. The calculation also shows that the aspect ratio and base angle have significant effect on flow and heat transfer in trapezoidal micro-channel.

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1. Introduction

In the past two decades, the rapid development of microelectromechanic systems (MEMS) and microchemical system has brought up a great interest in studying flow and heat transfer in micro-scale [1]. The Knudsen number $Kn$ can classify the gas flow in micro-channel into four flow regimes: continuum flow regime ($Kn < 0.001$), slip flow regime ($0.001 < Kn < 0.1$), transition flow regime ($0.1 < Kn < 10$) and free molecular flow regime ($Kn > 10$).

In order to understand the characteristics of fluid flow and heat transfer in microchannel, many researchers have conducted experimental investigations in recent years. Wu and Little [2] measured the...
friction factors of nitrogen, argon and helium flow in trapezoidal microchannels. The depth range from 130 to 200 $\mu m$ and the width was 30–60 $\mu m$. They found these friction factors were much larger than the predictions by the classic theory for laminar flow in conventional channels. Pfahler et al. [3,4] found that for microchannels with hydraulic diameters of 0.98–39.7 $\mu m$, and that the friction factors were lower than those predicted by the classic theory. Choi et al. [5] investigated the flow resistance of nitrogen gas flow in microtubes with hydraulic diameters between 3 and 81 $\mu m$, and found that the measured friction factors were smaller than those predicted by the classic theory.

Previous works [6] indicated that transport phenomena in the slip flow regime can be modeled by Navier-Stokes equations with boundary conditions that take into account the velocity slip and temperature jump at the wall. Theoretical and experimental analysis on slip flow in circular and rectangular microchannels has already been studied by many researchers [7,8]. However, slip flow in trapezoidal microchannels has not yet been investigated, in spite of its wide application in many Si-base MEMS [9].

The motivation of this paper is to study the fully developed laminar flow and heat transfer characteristics in the slip flow regime of trapezoidal microchannels. Through coordinate transformation, the governing equations on the computational domain are solved with uniform heat flux boundary conditions by a finite-difference scheme. The influences of velocity slip, temperature jump, and aspect ratio and base angle on friction coefficient and Nusselt number are investigated in detail.

2. Analysis

2.1. Problem statement and governing equations

We consider a Newtonian flow and heat transfer at slip flow regime in a trapezoidal microchannel. For the fully developed laminar steady state flow, ignoring body forces and viscous dissipation, the steady-state governing equations of momentum and energy can be written as

\[
\frac{\partial^2 u}{\partial \tilde{x}^2} + \frac{\partial^2 u}{\partial \tilde{y}^2} = \frac{D_h^2}{L \mu} \frac{\partial P}{\partial \tilde{z}}
\]

\[
\frac{\partial \Theta}{\partial \tilde{z}} = \frac{1}{Re Pr e} \frac{\tilde{u}}{u(\tilde{x}, \tilde{y})} \left( \frac{\partial^2 \Theta}{\partial \tilde{x}^2} + \frac{\partial^2 \Theta}{\partial \tilde{y}^2} \right).
\]

Where $Re$, $Pr$ is the Reynolds number and the Prandtl number, respectively, and the dimensionless variables are defined as

\[
\tilde{x} = x/D_h, \quad \tilde{y} = y/D_h, \quad \tilde{z} = z/L, \quad e = D_h/L.
\]

For the case of uniform wall heat flux, the dimensionless temperature is defined as

\[
\Theta = \frac{T - T_{in}}{qD_h/k}, \quad \Theta_n = 0, \quad \frac{\partial \Theta}{\partial \tilde{n}} \bigg|_{\text{wall}} = -1
\]
2.2. Slip-flow boundary conditions at the wall

The slip velocity and temperature jump boundary conditions to the first order are expressed as

\[ u_{\text{wall}} = -\beta_K n \frac{\partial u}{\partial n} \text{wall}, \quad \Theta_{\text{wall}} = -\beta_T Kn \frac{\partial \Theta}{\partial n} \text{wall} \]  \hspace{1cm} (5)

Where

\[ \beta_v = \frac{2 - \sigma_v}{\sigma_v}, \quad \beta_T = \frac{2 - \sigma_T}{\sigma_T} \frac{2\gamma}{\gamma + 1} \frac{1}{Pr}, \quad \beta = \frac{\beta_T}{\beta_v} \]  \hspace{1cm} (6)

In these equations, \( \sigma_v \) is the tangential momentum accommodation coefficient, and is considered to vary between 0 and 1, and \( \sigma_T \) is the thermal accommodation coefficient, which typically has a range from 0.01 to 1. The parameters, \( \beta_v \) and \( \beta_T \) indicate the influence of rarefaction and fluid-surface interaction. For typical engineering application, the value of the ratio \( \beta \) is close to 1.667 \([7,8]\). In the following of this paper we use parameter \( \beta \) instead of parameter \( \beta_T \).

2.3. Coordinate transformation

In order to convert the trapezoidal cross-section into a square cross-section, we transfer the independent variables on physical plane \((x, y)\) to a new set of independent variables in computational plane \((\xi, \eta)\) as shown in Fig. 1 where

\[ \xi = \frac{(a + x)H + (b - a)y}{2((b - a)y + aH)} \]  \hspace{1cm} (7)

\[ \eta = \frac{y}{H} \]  \hspace{1cm} (8)

With the above coordinative system transformation, governing equations on computational domain can be expressed as

\[ \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = C_1 \frac{\partial^2 \phi}{\partial \xi^2} + C_2 \frac{\partial^2 \phi}{\partial \xi \partial \eta} + C_3 \frac{\partial^2 \phi}{\partial \xi^2} + C_4 \frac{\partial \phi}{\partial \xi} + C_5 \frac{\partial \phi}{\partial \eta} \]  \hspace{1cm} (9)

\( C_1, C_2, C_3, C_4, C_5 \) are the coefficients in the transformed governing equation, and are obtained from

\[ C_1 = \alpha_1/J^2, \quad C_2 = \gamma_1/J^2, \quad C_3 = -2\beta_1/J^2, \quad C_4 = \xi''_y, \quad C_5 = 0 \]  \hspace{1cm} (10)

![Fig. 1. Transformation of a trapezoid on physical plane to a square on the computational plane.](image-url)
Where \( x_1, \beta_1, \gamma_1 \) and \( J \) are given by
\[
 x_1 = x_n^2 + y_n^2, \quad \beta_1 = x_n x_\xi + y_n y_\xi, \quad \gamma_1 = x_\xi^2 + y_\xi^2, \quad J = x_\xi y_\eta - y_\xi x_\eta.
\] (11)

The local normal gradient at the wall on computational domain is expressed as [10]
\[
 \frac{\partial \phi}{\partial \tilde{n}^{(n)}} = \frac{\gamma_1 \phi_\eta - \beta_1 \phi_\xi}{J \sqrt{\gamma_1}}, \quad \frac{\partial \phi}{\partial \tilde{n}^{(\xi)}} = \frac{x_1 \phi_\xi - \beta_1 \phi_\eta}{J \sqrt{\gamma_1}}.
\] (12)

2.4. Calculation of friction coefficient and Nusselt number

The main flow mean velocity and temperature in the cross-section are
\[
 \bar{u} = \frac{\int \int \bar{u} J d\xi d\eta}{\int \int J d\xi d\eta}, \quad \Theta_b = \frac{\int \int \Theta J d\xi d\eta}{\int \int \bar{u} J d\xi d\eta}.
\] (13)

The average temperature gradient at the wall is defined as
\[
 \frac{\partial \Theta}{\partial \tilde{n}} \bigg|_{\text{wall}} = \frac{\int_{I_1} \frac{\partial \Theta}{\partial \tilde{n}^{(n)}} \sqrt{x_1} d\eta + \int_{I_2} \frac{\partial \Theta}{\partial \tilde{n}^{(n)}} \sqrt{\gamma_1} d\xi + \int_{I_3} \frac{\partial \Theta}{\partial \tilde{n}^{(\xi)}} \sqrt{x_1} d\eta + \int_{I_4} \frac{\partial \Theta}{\partial \tilde{n}^{(n)}} \sqrt{\gamma_1} d\xi}{\int_{I_1} \sqrt{x_1} d\eta + \int_{I_2} \sqrt{\gamma_1} d\xi + \int_{I_3} \sqrt{x_1} d\eta + \int_{I_4} \sqrt{\gamma_1} d\xi}.
\] (14)

The local Fanning friction coefficient \( C_f = f \frac{\bar{u}}{\bar{u}} Re \) is obtained from
\[
 C_f = -\frac{D^2 \frac{\partial \bar{P}}{\partial \bar{z}}}{2L\mu \bar{u}}.
\] (15)

For the uniform wall heat flux boundary conditions, \( Nu_q \) is determined by
\[
 Nu_q = \frac{1}{\Theta_{\text{wall}} - \Theta_b}.
\] (16)

2.5. Numerical procedures

The finite-difference scheme is adopted to discrete the governing equation on the computational plane. A line-by-line method is used to solve the resulting algebraic equations. Alternating sweeps of...
tridiagonal matrix algorithm combined with a block correction is applied to each variable. Successive over-relaxation is also employed to improve convergence time.

3. Results and discussion

3.1. Friction coefficient

The effect of rarefaction on the friction coefficient \( C_F \) for different aspect ratios and base angles are shown in Figs. 2 and 3. As seen in these figures, \( C_F \) decreases with \( \beta_v Kn \) for fixed aspect ratio and base angle. Fig. 2 shows that \( \beta_v Kn \) has more significant effects on friction coefficient for channel with large aspect ratio. \( C_F \) increases with increasing value of aspect ratio when the aspect ratio is larger than 1, while it keeps constant when the aspect ratio is below 1. At high \( \beta_v Kn \), the influences of aspect ratio and base angle on friction coefficient become insignificant. Especially, when the aspect ratio is settled, the difference in \( C_F \) is very small for base angle bigger than 60°, see Fig. 3.

Fig. 4 shows the friction coefficient as a function of base angle for various aspect ratios. As seen in this figure, \( C_F \) increases with increasing of base angles. For ducts with aspect ratio larger than 1, a small base angle has more

Fig. 3. Variation of the fully developed Fanning friction coefficient \( C_F \) with \( \beta_v Kn \) for different base angle. Base angle: (1) 90°, (2) 75°, (3) 60°, (4) 45°; Aspect ratio: 1.

Fig. 4. Variation of the fully developed Fanning friction coefficient \( C_F \) with base angle for different aspect ratio. Aspect ratio: (1) 0.125, (2) 0.25, (3) 0.5, (4) 1.0, (5) 2.0, (6) 4.0, (7) 8.0; \( \beta_v Kn \): 0.05.
significant affects on friction coefficient, the influence of base angle on $C_F$ becomes more and more insignificant with the increase of base angle, when base angle is larger than $60^\circ$, the influence of base angle on $C_F$ can almost be neglected, and $C_F$ approaches asymptotically the value of that of the rectangular channel with same aspect ratio as the base angle increases. When the aspect ratio is below 1, there is little effect of the aspect ratio on friction coefficient for small base angle less than $60^\circ$, while for ducts with base angles larger than $60^\circ$ the friction coefficient changes more sharply with base angle and aspect ratio, $C_F$ quickly approaches the value of rectangular with same aspect ratio as the base angle increases.

3.2. Nusselt number

Figs. 5 and 6 show the effect of rarefaction and cross-section shapes on the developed Nusselt number when $\beta = 1.667$. It is worthwhile pointing out that $Nu_q$ and $C_F$ has similar trends for the same condition shown in Figs. 2–6.

Following to Larrode’s [7] and Yu’s [8] previous works on slip flow in circular tubes and in rectangular microchannels, we investigate the effect of rarefaction on the heat transfer in trapezoidal microchannel for
three case: $\beta=0.1$, $\beta=0.5$ and $\beta=1.667$. Fig. 7 shows the developed Nusselt number as function of aspect ratios for various base angles at $\beta=0.1$. It is found that $Nu_q$ increases with increasing $\beta, Kn$ for fixed aspect ratio, but rarefaction has less effect on $Nu_q$ for channels with large aspect ratios. Similar to the trend shown in Fig. 2, $Nu_q$ increase with aspect ratio for aspect ratios larger than 1, and remains constant for aspect ratios below 1. For $\beta=0.5$ (see Fig 8), when the aspect ratio smaller than 1, the effect of $\beta, Kn$ on Nusselt number is unobvious, while it become significant when the aspect ratio is higher than 1. When $\beta=1.667$, the results in Fig. 9 indicates a different trend from Fig. 7. These performances may be explained by considering the physical characteristics of velocity slip and temperature jump. Velocity slip increases the fluid velocity at the wall, enhancing convection heat transfer between the fluid and the wall; while temperature jump decreases temperature gradients at the wall, reducing the heat transfer between the fluid and the wall. When the value of $\beta$ is small, temperature jump is negligible compared with velocity slip, the heat transfer is controlled only by velocity slip, and so it is enhanced. For high $\beta$, the temperature jump is very significant, and it suppresses the enhancement of heat transfer due to velocity slip, so the Nusselt number decreases.

The results presented in Figs. 10 and 11 show the influence of $\beta$ on heat transfer as a function of aspect ratio and base angles. It is evident that convection heat transfer decreases with increasing the value of $\beta$. When $\beta$ is small, velocity slip enhances the heat transfer between the fluid and the wall; the developed Nusselt number is
larger than the value of no-slip flow. While for large \( \beta \), heat transfer is suppressed by the temperature jump at the wall; thus, Nusselt number is smaller than \( Nu_{\text{no-slip}} \). Specifically, as \( \beta \) increases, the \( Nu_q \) profile in Figs. 10 and 11 become more and more flat. The result reveals that the effects of aspect ratio and base angle on heat transfer are less obvious at large temperature jump.

4. Conclusions

Fully developed laminar flow and heat transfer for trapezoidal microchannel over the entire slip flow regime has been studied. Uniform wall heat flux boundary conditions were considered in this simulation. The friction coefficient in slip flow region was reduced in comparison with the no-slip flow results. The effects of aspect ratio and base angle on friction coefficient become less obvious at high \( \beta, Kn \). For small \( \beta \), convection heat transfer is controlled by velocity slip at the wall, the developed Nusselt number is larger than the value of no-slip flow. While for large \( \beta \), temperature jump dominates the heat transfer between the fluid and the wall, Nusselt number is smaller than
No-slip. The influences of aspect ratio and base angle on heat transfer are insignificant for large temperature jump.

**Nomenclature**

- $D_h$: hydraulic diameter (m)
- $L$: length of duct (m)
- $P$: pressure (Pa)
- $k$: thermal conductivity of fluid (W m$^{-1}$ K$^{-1}$)
- $q$: heat generation per unit area (W m$^{-2}$)
- $u$: axial velocity (m s$^{-1}$)
- $\bar{u}$: average velocity
- $\beta_v$: dimensionless variable defined in Eq. (6)
- $\beta_T$: dimensionless variable defined in Eq. (6)
- $\beta$: dimensionless variable $\beta = \beta_T/\beta_v$
- $\gamma$: specific heat ratio
- $\mu$: viscosity (Pa s)
- $\Theta$: dimensionless temperature Eq. (4)
- $\sigma_v$: tangential momentum accommodation coefficient
- $\sigma_T$: thermal accommodation coefficient
- $\varepsilon$: ratio of channel hydraulic diameter to its length
- $\Gamma$: boundaries of computational domain
- $\theta$: base angle (degree)

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